

Lagrangian Matroids Associated with Maps on Orientable Surfaces

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Introduction

The aim of this paper is to clarify the nature of combinatorial structures associated with maps on closed compact surfaces. These are fairly classical objects; however, not long ago it was discovered by A. Bouchet that maps are associated with Δ -matroids [7] (or Lagrangian matroids in the terminology of [4]). Δ -matroids are related to maps in almost the same way as ordinary matroids to graphs.

In this paper we explore this parallel further and show that the resulting Lagrangian matroids are representable by matrices in a sense analogous to representability of ordinary matroids, thus transferring the classical Rado's theorem [12] to Lagrangian matroids. The proper setting for the representation is provided by cohomology of the surface. Our proof is very elementary. It is worth mentioning that David Stone [13] found a cohomological proof of the main technical result of the paper, Theorem 5. His proof is not much shorter than ours but promises a possible generalisation of our result to higher dimension.

Our next observation is that the classical 'spanning tree algorithm' for graphs (which becomes the 'greedy algorithm' of matroid theory and the theory of Δ -matroids [6]) can be interpreted in our setting in a most elementary way, as a 'peeling off' procedure which cuts the (connected) surface into one closed ring-shaped peel, or, in some degenerate cases, to a 2-cell peel. This procedure is local, that is, at every step uses information only about a small part of the surface around the knife.

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1 Orthogonal matroids and maps

1.1 Maps on compact surfaces

A *graph* is a compact space G partitioned into a finite set $E = E(G)$ of open 1-cells called *edges* and a finite set $V = V(G)$ of points called *vertices* such that every edge has a vertex at each end (possibly the same vertex at both ends, in which case we say that the edge is a loop). If S is a compact surface and $G \subseteq S$ we say that the graph is drawn on the surface S . We shall be concerned only with orientable connected compact surfaces in this paper. If connected components of $S \setminus G$ are open 2-cells, then we say that $\mathcal{M} := (G, S)$ is a *map*, and the connected components of $S \setminus G$ are called the *faces* or *countries* of the map. Thus a map \mathcal{M} introduces on S a structure of a 2-dimensional cell complex. We assume that the set of edges (correspondingly, coedges) of \mathcal{M} is non-empty, excluding from consideration a trivial map on a sphere with one vertex and no edges.

A dual map $\mathcal{M}^* = (G^*, S)$ is defined in the usual way. Its vertices are in one-to-one correspondence with the faces of \mathcal{M} ; for each face of \mathcal{M} , we choose a point inside it as the corresponding vertex of \mathcal{M}^* . The edges are chosen such that every edge e of G is disjoint from every edge of G^* except for its dual edge, e^* , which has with e exactly one point in common. We call the e^* the coedges. Then it can be shown that the faces of \mathcal{M}^* correspond to the vertices of \mathcal{M} : every face of \mathcal{M}^* contains a unique vertex of \mathcal{M} , and every vertex of \mathcal{M} belongs to a unique face of \mathcal{M}^* . The vertices of \mathcal{M}^* are called *covertices* and the set of covertices is denoted by V^* . Finally, notice that \mathcal{M} is a dual map to \mathcal{M}^* .

For $F \subseteq E$ we define $F^* := \{e^* \mid e \in F\}$ and $\overline{F} := E \setminus F$. We call a set $F \subset E \cup E^*$ *admissible* if $F \cap F^* = \emptyset$. It will be convenient to index edges in E by elements of $I = \{1, \dots, n\}$ and the coedges in E^* by the corresponding elements of I^* , so that

$$E = \{e_1, \dots, e_n\} \text{ and } E^* = \{e_{1^*}, \dots, e_{n^*}\}.$$

1.2 Symplectic and orthogonal matroids

For a fuller exposition of the theory and definitions of symplectic matroids, see [4] and the forthcoming book [5]. Let

$$I = \{1, 2, \dots, n\} \text{ and } I^* = \{1^*, 2^*, \dots, n^*\}$$

and $J = I \cup I^*$. Define maps $^* : I \rightarrow I^*$ by $i \mapsto i^*$ and $^* : I^* \rightarrow I$ by $i^* \mapsto i$, so that * is an involutive permutation of J . Then we say that a subset $K \subset J$ is *admissible* if and only if $K \cap K^* = \emptyset$. We denote by J_k the set of all admissible k -subsets of J .

Let $\mathcal{B} \subseteq J_n$ be a set of admissible n -subsets of J and let M be the triple $(J, ^*, \mathcal{B})$. Then M is a *Lagrangian matroid* if it satisfies the **Symmetric Exchange Axiom**:

For any $A, B \in \mathcal{B}$ and $k \in A \triangle B$ there exists some $i \in A \triangle B$ such that $A \triangle \{k, k^*, i, i^*\} \in \mathcal{B}$.

Here \triangle is the symmetric difference; $A \triangle B := (A \cup B) \setminus (A \cap B)$. This axiom is due to Bouchet [6], Dress and Havel [8, 11], where Lagrangian symplectic matroids appeared, cryptomorphically, under the names of Δ -matroids or *metroids*. We call \mathcal{B} the set of *bases* of M . A Lagrangian matroid is a special case of a symplectic matroid; in a general symplectic matroid, the bases are elements of J_k for some k . An appropriate axiom system is given in [4]. In this paper, we shall only be concerned with Lagrangian

matroids. An *orthogonal matroid* or *even matroid* is a symplectic matroid in which the difference between the number of starred elements in any two bases is always even.

We say that an admissible set $F \subset E \cup E^*$ is *independent* if $S \setminus cl(F)$ is connected. A *basis* of M is a maximal independent admissible set.

Bouchet [7] proved that the set $\mathcal{B} = \mathcal{B}(M)$ of all bases of a map M is a even Lagrangian matroid on the set $E \cup E^*$. This paper provides another proof of Bouchet's result with a important improvement of *representability*; necessary terms are explained in Subsection 1.4.

Theorem 1 *Let \mathcal{M} be a map with n edges on a orientable compact closed surface S . Then*

- *all bases of \mathcal{M} have cardinality n and*
- *the set \mathcal{B} of all bases is an orthogonally representable over \mathbb{Q} , orthogonal Lagrangian matroid.*

It is well-known that (ordinary) matroids associated with graphs are representable by matrices ([12], the exposition can be found in any book on matroid theory, see, for example, [15]). Our Theorem 1 is a generalisation of this result. We shall see in the next subsection that the classical Spanning Tree Algorithm for graphs, which corresponds to the Greedy Algorithm of matroid theory, also has a natural analogue for maps.

1.3 The greedy algorithm and peeling the skin.

Draw on each face of M the segments connecting the covertex with the vertices of this face, we shall call them *diagonals*. Edges, coedges and diagonals define a baricentric subdivision of M . Each of the triangles of this subdivision has one side a diagonal, one side a half-edge of the map, and one side a half-coedge. It can be immediately seen that, when cutting along the edges and coedges of a basis, we obtain a ring (topologically, a punctured 2-cell) or a 2-cell. This is due to the fact that, in view of Theorem 1, the basis forms an admissible set of n elements, and thus every triangle of the baricentric subdivision is cut along precisely one edge. This means that each triangle has two neighbors in the modified surface; hence it is a ring or a 2-cell.

The procedure for peeling is the following. We construct a sequence of triangles S_0, S_1, \dots, S_n in the baricentric subdivision in which S_i and S_{i-1} , $i = 1, 2, \dots$, are adjacent, i.e. have in common an edge, coedge or diagonal.

PEELING OFF PROCEDURE

0° Start at arbitrary triangle S_0 in the subdivision.

1° Assume that we have constructed S_0, S_1, \dots, S_i .

- If neither edge e nor coedge e^* which bound S_i has been cut along at the previous steps of the procedure, cut the surface along the entire edge e or coedge e^* choosing e or e^* so that the surface remains connected. Take for S_{i+1} the triangle which has a common with S_i coedge (correspondingly, edge) which was not cut.
- If either edge e or coedge e^* has been cut along at one of the previous steps of the procedure, take for S_{i+1} the triangle which lies across the diagonal or noncut (co)edge from S_i and is different from S_{i-1} .

2° If step 1° cannot be made or if S_{i+1} defined by rule 1° is one of the triangles S_0, S_1, \dots, S_{i-1} , stop.

We may reformulate this description in terms of independent sets. By definition, an independent set of (co)edges is characterised by the property that it is admissible and its closure does not disconnect the surface. Now the condition of Step 1° is that we must make a cut which leaves the set of cuts made an independent set. By Theorem 1 maximal independent sets (*bases*) have cardinality n and form a Lagrangian matroid on $E \cup E^*$. Notice that, due to our definition of a basis of a map, independent subsets of (co)edges are just subsets of bases.

We use a simplified version of the greedy algorithm for Lagrangian matroids as described by the following procedure.

Let \mathcal{B} be the set of bases of a Lagrangian matroid on J . Let I be the set of all independent subsets, that is all subsets of bases from \mathcal{B} . Let i_1, \dots, i_n be the elements $1, \dots, n$ written in some order $i_1 \prec \dots \prec i_n$.

GREEDY ALGORITHM

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 $B := \emptyset;$ 
for  $j = 1, 2, \dots, n$  do
  if  $B \sqcup \{i_j\} \in I$  then  $B := B \sqcup \{i_j\}$  else  $B := B \sqcup \{i_j^*\};$ 
end

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Since every basis of a Lagrangian matroid contains, for every $i = 1, \dots, n$, one of the elements i, i^* , it is immediately obvious that the greedy algorithm returns a basis in \mathcal{B} .

A stronger version of a greedy algorithm, due to Bouchet [6], can be used for characterisation of Lagrangian matroids; we do not use it here.

The greedy algorithm allows us to prove that the peeling-off procedure described produces the set of cuts which is a basis of the map.

Let us pause the procedure at some stage when we have chosen the triangles S_0, S_1, \dots, S_k and let the set of cuts already made be B . Notice that B is an independent set. If S_k has a cut (co)edge we have to select for S_{k+1} the triangle lying across the diagonal or non-cut (co)edge from S_k ; we can do this unless S_{k+1} is one of the triangles S_0, \dots, S_k ; but in this case one can immediately see that $S_{k+1} = S_0$ and we cut a closed ribbon from the surface. Since cuts along B do not disconnect the surface, the ribbon covers the whole surface.

Hence we can assume that neither edge e_i nor coedge e_{i^*} bounding the triangle S_k has not been cut yet. We may define an ordering \prec on $\{1, \dots, n\}$ in which non-starred elements in $B \cup B^*$ precede all other elements in $\{1, \dots, n\}$. Obviously, the greedy algorithm for \prec will at some stage produce B . Furthermore, we may assign i to immediately succeed B in the ordering \prec , then the greedy algorithm says that we can cut along one of the edges e_i or e_{i^*} retaining the surface connected, and thus continue the process as desired.

Therefore we have proven the following result.

Theorem 2 *Every simplex in the triangulation of the map $M = (S, G)$ appears in the sequence S_0, S_1, \dots, S_n exactly once. The surface \bar{S} obtained by all cuts is homeomorphic to a ring (punctured 2-cell) or 2-cell.*

1.4 Representability of Lagrangian matroids

Let V be the vector space over a field K of characteristic $\neq 2$ whose basis is $\{e_i, e_i^* \mid i \in I\}$. Define a symmetric bilinear form on V by $\langle e_i, e_i^* \rangle = 1$, $\langle e_i, e_j \rangle = 0$ for all $i \in I$ and $j \in J$ with $i^* \neq j$, so that the basis $\{e_i, e_i^* \mid i \in I\}$ is a *hyperbolic basis* in V . L is a *Lagrangian* subspace of V if it is a totally isotropic subspace (that is, $\langle k, l \rangle = 0$ for every $k, l \in L$) of maximal dimension. If f_1, \dots, f_n is a basis in a n -dimensional subspace L in V , then we can associate with it a $n \times 2n$ matrix C written as (A, B) for two $n \times n$ matrices A and B such that

$$f_i = \sum_{j=1}^n a_{ij} e_j + \sum_{j=1}^n b_{ij} e_j^*.$$

It is easy to see that L is a Lagrangian subspace in V if and only if AB^t is a skew symmetric matrix.

Let us index the columns of A with I and those of B with I^* , so that the columns of C are indexed by J . We say that an admissible subset $F \subseteq J$ is *independent* if the corresponding columns of C are linearly independent. Define the set of *bases* $\mathcal{B} \subseteq J_n$ by putting $X \in \mathcal{B}$ if and only if

- $X \in J_n$ and
- the $n \times n$ minor consisting of the i -th column of C for all $i \in X$ is non-zero.

Notice that change of a basis in L is equivalent to conducting row operations on the matrix (A, B) and leaves the pattern of dependencies of the columns unchanged. Therefore the set \mathcal{B} depends only on L and not on choice of basis in L .

Theorem 3 (A. Vince and N. White [14]) *If L is a Lagrangian subspace in V then*

- *every independent subset in J belongs to a basis in \mathcal{B} , and*
- *\mathcal{B} is the set of bases of an orthogonal Lagrangian matroid.*

For a proof that the axioms used to define Lagrangian matroids in [14] are equivalent to the Symmetric Exchange Axiom, see Wenzel [16] or the book [5].

We call an orthogonal matroid arising from a Lagrangian subspace L written by a matrix (A, B) with AB^t skew-symmetric an *orthogonally representable* orthogonal matroid, and say that (A, B) is an *orthogonal representation* of it over the field K .¹

2 Matroids, Representations and Maps

Let \mathcal{M} be a map on a compact connected orientable surface S , with the edge and coedge sets $E = \{e_i \mid i \in I\}$ and $E^* = \{e_i^* \mid i \in I^*\}$, $I = \{1, \dots, n\}$, vertex set V and covertex set V^* . We orient edges $e \in E$ in an arbitrary way and choose orientation of coedges e^* so that the intersection index $(e, e^*) = 1$ for all $e \in E$.

We shall use homology of S and $S \setminus (V \cup V^*)$ with coefficients in \mathbb{Q} .

¹Note in passing that there is also another way of representing Lagrangian matroids, by Lagrangian subspaces in a symplectic vector space [4]. In the more general setting of Coxeter matroids [3] the fact that every orthogonal Lagrangian matroid is a symplectic Lagrangian matroid is explained by the canonical embedding of the corresponding Coxeter groups $D_n < C_n$ and an observation that the thick D_n -building of isotropic subspaces in an orthogonal space has the natural structure of a thin C_n -building [9].

For a cycle c in $H = H_1(S \setminus (V \cup V^*))$ we have the well-defined intersection index, denoted here by (c, e) , of c with an edge (or coedge) $e \in E \cup E^*$. We shall denote by \hat{e} the corresponding linear functional $\hat{e} : c \mapsto (c, e)$ from H^* . Thus \hat{e} is a cocycle in $H^1(S \setminus (V \cup V^*))$. Notice that this $H^1(S \setminus (V \cup V^*))$ can be identified with $H_1(S, V \cup V^*)$ by Poincare-Lefschetz duality [10, VIII.7].

Lemma 4 *An admissible set of (co)edges $F \subset E \cup E^*$ is independent if and only if the linear functionals \hat{f} , $f \in F$, are linearly independent over \mathbb{Q} .*

Proof Denote $X = V \cup V^* \cup \bigcup_{f \in F} f$. Then we have the triple $W \subset X \subset S$ of cell complexes. The lemma immediately follows from the long exact homological sequence for this triple:

$$\begin{aligned} \cdots \longrightarrow H_2(X, V \cup V^*) &\xrightarrow{\alpha_2} H_2(S, V \cup V^*) \xrightarrow{\beta_2} H_2(S, X) \\ &\xrightarrow{\partial_2} H_1(X, V \cup V^*) \xrightarrow{\alpha_1} H_1(S, V \cup V^*) \longrightarrow \cdots \end{aligned}$$

Since $H_2(X, V \cup V^*) = 0$, β_2 is an injection. If $S \setminus X$ is connected then $H_2(S, X)$ is 1-dimensional, and, from comparing dimensions, we see that β_2 is a surjection. Hence $\partial_2 = 0$ and α_1 is an injection. We need to notice only that $H_1(X, V \cup V^*)$ is generated by (co)edges in F . \diamond

Introduce the vector space $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$ over \mathbb{Q} with the basis $E \cup E^*$.

Define a symmetric bilinear form on $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$ by $\langle e, e^* \rangle = 1$, and $\langle e, f \rangle = 0$ for all $e, f \in E \cup E^*$ such that $e^* \neq f$, so that $E \cup E^*$ is a hyperbolic basis in $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$.

For $c \in H$, the *incidence vector* of c , denoted $\mathfrak{t}(c) \in \mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$, is defined by

$$\mathfrak{t}(c) = \sum_{e \in E \cup E^*} (c, e)e.$$

The main technical result of the paper is:

Theorem 5 *The image $\mathfrak{t}(H)$ of $H = H_1(S \setminus (V \cup V^*))$ under the map $c \mapsto \mathfrak{t}(c)$ is an isotropic subspace of the orthogonal space $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$.*

We postpone the proof of Theorem 5 until the next section, and meanwhile deduce from it the main result of the paper, Theorem 1.

Theorem 6 *The isotropic subspace $\mathfrak{t}(H)$ is Lagrangian, and the Lagrangian orthogonal matroid corresponding to $\mathfrak{t}(H)$ has bases which correspond to the bases of the map \mathcal{M} .*

Proof Notice first that the map $\mathcal{M} = (G, S)$ has at least one basis of cardinality n . To construct it, take a spanning tree T in the graph G (T can be empty), then $B = T \cup \overline{T}^*$ is rather obviously a basis of \mathcal{M} and has cardinality n [7]. If now b_1, \dots, b_n are the (co)edges in B then the linear functionals $\hat{b}_1, \dots, \hat{b}_n$ on H are linearly independent, hence the n functionals \tilde{b}_i on $\mathfrak{t}(H)$ defined by the rule

$$\tilde{b}_i(\mathfrak{t}(c)) = \hat{b}_i(c)$$

are also linearly independent. Therefore $\dim \mathfrak{t}(H) \geq n$, and being an isotropic subspace in $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$, it is a Lagrangian subspace. Notice that if we represent $\mathfrak{t}(H)$ by a

$n \times 2n$ -matrix M in the basis $e_1, \dots, e_n, e_1^*, \dots, e_n^*$, the columns of M will represent the functionals $\tilde{e}_i, \tilde{e}_i^*$. By Lemma 4 the columns of M are linearly independent if and only if the correspondent set of (co)edges is independent in \mathcal{M} . Hence the bases of the Lagrangian matroid associated with $\mathfrak{t}(H)$ are exactly the bases of the map \mathcal{M} . \diamond

Now Theorem 1 is an immediate corollary of Theorem 6.

3 Proof of Theorem 5

We shall prove Theorem 5 by a naive geometric argument: we shall gradually simplify the map M and use induction on the total number of vertices and covertices in the map. David Stone [13] offered a cohomological proof of the same result.

REDUCTION, STEP 1. Notice that if an edge $e \in E$ is a contractible loop, i.e. the endpoints of e coincide and the closure $cl(e)$ is a contractible cycle on S then $\langle c, e \rangle = 0$ for every $c \in H$ and thus the contribution

$$(c, e)(d, e^*) + (c, e^*)(d, e)$$

of e -th and e^* -th coordinates of the incidence vectors $\mathfrak{t}(c)$ and $\mathfrak{t}(d)$ to their scalar product $\langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle$ is 0. Hence we can assume without loss of generality that the graph G (and, analogously, the dual graph G^*) contains no contractible loops.

REDUCTION, STEP 2. Introduce the scalar product $\langle \cdot, \cdot \rangle$ on H by setting

$$\langle c, d \rangle = \langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle.$$

Our aim is to prove that $\langle \cdot, \cdot \rangle$ vanishes on H .

Consider the kernel K of the canonical map

$$H_1(S \setminus (V \cup V^*)) \longrightarrow H_1(S).$$

It is well-known that K is isomorphic to $H_0(V \cup V^*)$ and generated by small cycles around vertices and covertices in $V \cup V^*$.

Lemma 7 $\langle K, H \rangle = 0$.

Proof Indeed, let c be a small cycle around a covertex $v^* \in V^*$ (for vertices the proof is analogous). Let e_1^*, \dots, e_k^* be the oriented edges exiting from or entering v ; notice that we count loops with both ends at v^* twice, but with opposite signs. The corresponding edges e_1, \dots, e_k form the boundary of the country C around v ; again, edges corresponding to loops e_i^* appear twice, with opposite orientations. Now if d is an arbitrary cycle in H then

$$\langle c, d \rangle = \langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle = (c, e_1^*)(d, e_1) + \dots + (c, e_k^*)(d, e_k).$$

Without loss of generality we can change the orientation of edges and co-edges in such a way that $(c, e_i^*) = 1$ for all coedges $e_i^*, i = 1, \dots, k$ which are not loops and $e_1 + \dots + e_k$ is the boundary cycle ∂C of the country C around v^* . Then the right hand side of the previous equation is exactly the index of intersection of d and ∂C , and hence equals 0. \diamond

Lemma 7 allows us to transfer the scalar product $\langle \cdot, \cdot \rangle$ from $H_1(S \setminus (V \cup V^*))$ to $H_1(S)$: if $c, d \in H_1(S)$ and c', d' are any preimages of c and d in $H_1(S \setminus (V \cup V^*))$, then we set $\langle c, d \rangle = \langle c', d' \rangle$. The theorem which we are proving amounts to saying that $\langle c, d \rangle = 0$ for all $c, d \in H_1(S)$. We can work with cycles in $H_1(S)$ instead of $H_1(S \setminus (V \cup V^*))$ which gives us the necessary degree of flexibility in geometric construction we shall use in the proof.

Now we can start the induction on $|V \cap V^*|$.

BASIS OF INDUCTION. For the basis of induction assume that the map \mathcal{M} and the dual map \mathcal{M}^* both contain exactly one vertex, v and v^* , correspondingly. The orientable connected compact surface S is a sphere with m handles. Then there are $n = 2m$ edges and $2m$ coedges, from the Euler formula. We can treat the edges e_i and coedges e_i^* , $i \in I = \{1, \dots, n\}$ of the map as elements of $H_1(S)$. Consider again the kernel K of the canonical surjective map

$$H_1(S \setminus (V \cup V^*)) \longrightarrow H_1(S);$$

K is generated by two cycles k_1 and k_2 around v and v^* . It is easy to see that $\mathfrak{t}(k_1) = \mathfrak{t}(k_2) = 0$, hence $K \leq \ker \mathfrak{t}$ and \mathfrak{t} can be lifted to a linear map from $H_1(S)$ to $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$ which we denote by the same symbol \mathfrak{t} . Thus $\langle c, d \rangle = \langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle$ for $c, d \in H_1(S)$ and in our computations we can work with elements of $H_1(S)$ instead of $H_1(S \setminus (V \cup V^*))$.

The intersection index

$$H_1(S) \times H_1(S) \longrightarrow H_0(S) = \mathbb{Q}$$

is a non-degenerate skew symmetric form on $H_1(S)$. From the definition of orientation of co-edges we have

$$(e_i, e_j^*) = \delta_{ij} \text{ for } i, j \in I.$$

Since $\dim H_1(S) = 2m = |I|$, it follows from elementary linear algebra that $\{e_i\}$ and $\{e_i^*\}$ are dual bases in $H_1(S)$.

If $c \in H_1(S)$ then we may express $c = \sum_{i \in I} c_i e_i$, for some coefficients c_i . Now we have $\mathfrak{t}(c)$ expressed as a sum of $\mathfrak{t}(e_i)$. So, if we can show that the space generated by the $\mathfrak{t}(e_i)$ is totally isotropic, we have completed the proof. Now, from the definition of an incidence vector,

$$\mathfrak{t}(e_i) = \sum_{j \in I} ((e_i, e_j) e_j + (e_i, e_j^*) e_j^*).$$

Now, taking $i, k \in I$, we have, since $(e_i, e_j^*) = \delta_{ij}$,

$$\begin{aligned} \langle \mathfrak{t}(e_i), \mathfrak{t}(e_k) \rangle &= \left\langle \left(\sum_{j \in I} (e_i, e_j) e_j + e_i^* \right), \left(\sum_{j \in I} (e_k, e_j) e_j + e_k^* \right) \right\rangle \\ &= \left\langle \sum_{j \in I} (e_i, e_j) e_j, e_k^* \right\rangle + \left\langle e_i^*, \sum_{j \in I} (e_k, e_j) e_j \right\rangle \\ &= (e_i, e_k) + (e_k, e_i) \\ &= 0. \end{aligned}$$

Thus $\mathfrak{t}(H)$ is generated by a set of elements with zero pairwise scalar products, so the theorem follows.

THE INDUCTIVE STEP. Assume that we have a map \mathcal{M} with $|V \cup V^*| > 2$. Then \mathcal{M} has a contractible edge or coedge, say, edge f . Let \mathcal{M}' be a map on the same surface S obtained by contracting the edge e (in the way shown on Figures 1 and 2) V' and V'^* its sets of vertices and covertices. We identify E' with $E \setminus \{f\}$ and E'^* with $E^* \setminus \{f^*\}$, correspondingly, and $\mathbb{Q}^{E'} \oplus \mathbb{Q}^{E'^*}$ with a subspace of $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$, so that the scalar product \langle, \rangle on $\mathbb{Q}^{E'} \oplus \mathbb{Q}^{E'^*}$ is the restriction of the scalar product on $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$. The map \mathcal{M}' defines the linear map

$$\iota' : H_1(S \setminus (V' \cup V'^*)) \longrightarrow \mathbb{Q}^{E'} \oplus \mathbb{Q}^{E'^*},$$

which allows to lift the scalar product from $\mathbb{Q}^{E'} \oplus \mathbb{Q}^{E'^*}$ to scalar products \langle, \rangle' on $H_1(S)$ and $H' = H_1(S \setminus (V' \cup V'^*))$.

Lemma 8 *Contracting a contractable edge f does not affect the scalar product \langle, \rangle in the following sense: given two cycles $c, d \in H$ we can contract f in such way that, for the new map \mathcal{M}' and appropriate $c', d' \in H'$,*

$$\langle c, d \rangle = \langle c', d' \rangle'.$$

Proof Figures 1 and 2 depict the process of contraction.

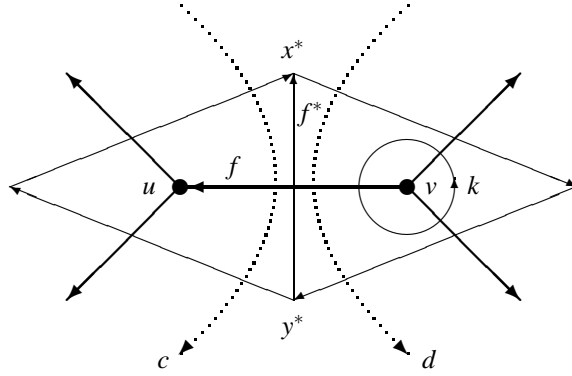
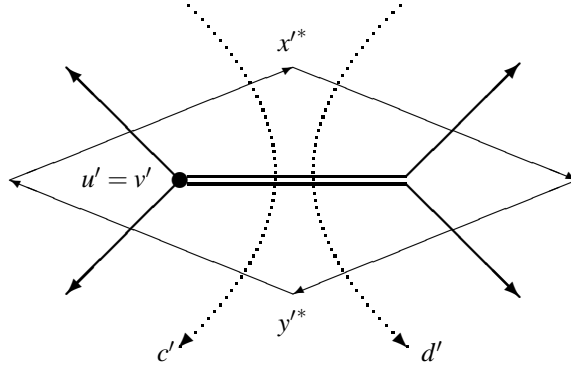


Figure 1: Fragment of the map \mathcal{M} before contraction.

When contracting the edge f we wish to keep all changes restricted to the closure of the union of two countries of the dual map \mathcal{M}^* containing the endpoints u and v of the contracted edge f . Figure 1 is pre-contraction and Figure 2 is post-contraction. The horizontal edge $f = (uv)$ has been contracted to a point $u' = v'$ combining the vertices at its ends, and the corresponding coedge $f^* = (x^*y^*)$ has disappeared. The cycles c' and d' are canonical images of c and d in $H_1(S \setminus (V' \cup V'^*))$.

As a result, c' , in comparison with c , has lost its intersections with the coedge f^* and edge f but has attained intersections with every edge $e \neq f$ which had exited from v prior to contraction; notice, however, that the sign of the index of intersection has changed:

$$(c, e) = -(c', e).$$

Figure 2: Map \mathcal{M}' after contraction.

Hence

$$\begin{aligned} \mathfrak{t}'(c') &= \mathfrak{t}(c) - (c, f^*)f^* - (c, f)f - (c, f) \cdot \sum_{e \text{ exits from } v, e \neq f} e \\ &= \mathfrak{t}(c) - (c, f^*)f^* - (c, f) \cdot \sum_{e \text{ exits from } v} e. \end{aligned}$$

If k is a small circle around v , as shown on Figure 1, then

$$\sum_{e \text{ exits from } v} e = \mathfrak{t}(k)$$

and

$$\mathfrak{t}'(c') = \mathfrak{t}(c) - (c, f^*)f^* - (c, f)\mathfrak{t}(k).$$

Analogously

$$\mathfrak{t}'(d') = \mathfrak{t}(d) - (d, f^*)f^* - (d, f)\mathfrak{t}(k).$$

Notice that

$$\langle \mathfrak{t}(c), \mathfrak{t}(k) \rangle = \langle \mathfrak{t}(d), \mathfrak{t}(k) \rangle = \langle \mathfrak{t}(k), \mathfrak{t}(k) \rangle$$

by Lemma 7 and $\langle f^*, f^* \rangle = 0$ by definition of the scalar product $\langle \cdot, \cdot \rangle$. Now we can compute:

$$\begin{aligned} \langle c', d' \rangle' &= \langle \mathfrak{t}(c) - (c, f^*)f^* - (c, f)\mathfrak{t}(k), \mathfrak{t}(d) - (d, f^*)f^* - (d, f)\mathfrak{t}(k) \rangle \\ &= \langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle - (d, f^*)\langle \mathfrak{t}(c), f^* \rangle - (c, f^*)\langle f^*, \mathfrak{t}(d) \rangle \\ &\quad + (c, f^*)(d, f)\langle f^*, \mathfrak{t}(k) \rangle + (c, f)(d, f^*)\langle \mathfrak{t}(k), f^* \rangle. \end{aligned}$$

But

$$\langle \mathfrak{t}(c), f^* \rangle = (c, f), \quad \langle f^*, \mathfrak{t}(d) \rangle = (d, f), \quad \langle f^*, \mathfrak{t}(k) \rangle = 1,$$

hence we can simplify:

$$\begin{aligned} \langle c', d' \rangle' &= \langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle - (d, f^*)(c, f) - (c, f^*)(d, f) \\ &\quad + (c, f^*)(d, f) + (c, f)(d, f^*) \\ &= \langle \mathfrak{t}(c), \mathfrak{t}(d) \rangle \\ &= \langle c, d \rangle. \end{aligned}$$

◇

Since contraction of an edge (coedge) does not change the scalar product \langle, \rangle on $H_1(S)$, we can complete the proof of Theorem 5 by induction on $|V \cup V^*|$.

4 Non-orientable surfaces

Obviously in the case when S is a not necessary orientable compact surface, we can make all homological computations modulo 2. Symmetric scalar product in that case is also skew symmetric, and, using obvious modifications in terminology and notation, one can prove the following two results.

Theorem 9 *The image $\mathfrak{t}(H)$ of $H = H_1(S \setminus (V \cup V^*))$ under the map $c \mapsto \mathfrak{t}(c)$ is an isotropic subspace of the symplectic space $\mathbb{F}_2^E \oplus \mathbb{F}_2^{E^*}$.*

Theorem 10 *If \mathcal{M} is a map on a compact surface S then the set \mathcal{B} of its bases is a representable over \mathbb{F}_2 symplectic Lagrangian matroid.*

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